A General Theory for the Flow of Weakly Ionized Gases

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A general theory is developed for the flow of a weakly ionized gas about an arbitrary solid body with absorbing surfaces The main interest lies in the prediction of the electrical responses of the body as a function of the pertinent properties of the flow The theory is based on continuum formulation, and is valid when 1) the mean-free-path of the charged particles is much smaller than the thickness of the sheath and 2) the debye length is much smaller than the thickness of the boundary layer adjacent to the body surface The entire range of flow velocity in terms of an electric Reynolds number R is investigated, but the detailed analysis is devoted to the case $R^{1/2} \gg 1$ It is found that the electrical disturbances can be divided into three physically distinct and mathematically uncoupled regions, namely the outer region, the ambipolar diffusion region, and the sheath region Closed-form analytical results are obtained for the floating potential and the current-voltage characteristic These are useful in the interpretations of Langmuir probe data Detailed structures of the solutions are given in terms of explicit universal functions

I Introduction

THE importance of Langmuir probes as a research tool in ionized gases is well recognized The basic theory for such probes, however, remains rather inadequate In most applications the density is extremely low so that continuum formulations are difficult to justify However, because of the inherent simplicity of continuum in comparison with molecular theory, it seems reasonable to attack the problem by investigating thoroughly the consequences of a continuum theory and see if various kinds of rarefied corrections can be developed later In any case, the presence of a correct continuum theory should help improve our understanding of the problem in the entire density range

In the present paper, we are concerned with the flow of a weakly ionized gas over an arbitrary solid body and are interested in the electrical response of the body When the gas is quiescent and not flowing, the continuum theory for a sphere has been treated extensively by Su and Lam[1] and Cohen [2] When the gas is flowing, the problem near the stagnation point has been treated by Chung[3] and Talbot [4]

The basic assumptions underlying the present theory are the following:

1) The mean-free-path l of the charged particles is very small and is much smaller than the thickness of the sheath adjacent to the body surface

2) The characteristic length L of the problem is much larger than h , the debye length We require that

$$\frac{h_e}{L} \ll R^{-1/2}$$

where R is the electric Reynolds number, which is generally of the same order as the viscous Reynolds number

3) The Mach number of the flow is very small so that the flow can be assumed incompressible

4) The diffusion velocities of the ions and electrons due to any electric field present are small in comparison with their thermal velocities

5) The gas is only weakly ionized so that no electrohydrodynamic interactions are considered The velocity field of the neutrals is assumed known

6) The flow is steady; the freestream is neutral with zero electric potential and uniform charge densities The processes of ionization and de-ionization are frozen in the gas phase There is no applied magnetic field, and induced magnetic effects are neglected

7) All physical properties such as diffusion coefficients are taken to be constants

8) Electron and ion temperatures may have different values but are uniform in space

9) The solid surface is "absorbing "

Within this much simplified framework, the general problem can readily be formulated The physical phenomena being studied are as follows Whenever a charged particle strikes an "absorbing" surface, the present theory stipulates that it is absorbed in the sense that it loses its charge by recombination on the surface Thus, solid surfaces act as sinks to charged particles Electrons, in general, have much larger thermal velocities than the massive ions, and, consequently, per unit time more electrons are likely to strike the surface than the slower moving ions As the electrons diffuse in the general direction of the surface, the slower ions retard the diffusion by setting up an electrostatic field This process is called ambipolar diffusion, and the associated electric potential field falls in the direction of the charge motion Immediately next to the body surface the number density of electrons becomes too low to carry the ions, and the potential of the body surface and the ions' own diffusion motion take over A sheath of high-electric field, therefore, exists Outside the ambipolar diffusion region, convection effects dominate, and the electron and ion number densities are quite uniform However, for $R^{-1/2} \ll 1$, the effects of the body potential may penetrate far outside the ambipolar region, giving rise to currents Thus, the mere presence of solid surfaces in an ionized gas will, in general, cause electrical disturbances It is the purpose of the present paper to study these electrical effects

2 General Formulation

We consider an incompressible viscous flow over an arbitrary solid body The velocity field $\mathbf{q}$ of the neutral gas is

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assumed known The electrostatic problem is then governed by the following set of equations:

$$\nabla^2 P = -4\pi e L^2 (Zn_i - n) \tag{2 1}$$

$$\mathbf{q} \ \nabla n_i + \nabla \ \mathbf{\Gamma}_i = 0 \tag{2 2}$$

$$\mathbf{q} \ \nabla n + \nabla \ \mathbf{\Gamma} = 0 \tag{2 3}$$

$$\mathbf{\Gamma}_i = \frac{1}{L}\left[-D_i \nabla n_i - \frac{eZD_i}{kT_i} n_i \nabla P\right] \tag{2 4}$$

$$\mathbf{\Gamma} = \frac{1}{L}\left[-D \nabla n + \frac{eD_e}{kT} n \nabla P\right] \tag{2 5}$$

where P is electric potential, L is a characteristic length with which all spatial variables are nondimensionalized, Z is the charge number of ions, n_i and n are number densities, $\mathbf{\Gamma}_i$ and $\mathbf{\Gamma}$ are number fluxes, D_i and D are diffusion coefficients, and T_i and T are temperatures for ions and electrons, respectively It is assumed that T_i need not be identical with T, and the ratio T_i/T is at most $O(1)$ Mobilities of ion and electrons have been related to their diffusion coefficients by means of Einstein's relation All physical properties are assumed constant, and the gas is assumed to be weakly ionized with constant electron number density n_∞ in the freestream We shall consider only steady flow problems

We nondimensionalize the variables as follows:

$$\begin{aligned} \mathbf{q} &= U_\infty \mathbf{V} \qquad \psi = -e\psi/(kT) \\ \mathbf{\Gamma}_i &= \gamma_i D_i n_\infty/(\epsilon Z l) \\ \mathbf{\Gamma} &= \gamma D n_\infty / L \\ N_i &= Zn_i/n_\infty \quad N = n/n_\infty \end{aligned} \tag{2 6}$$

where U_∞ is the characteristic velocity of the problem We denote by α, β, ϵ, and R, the following dimensionless parameters:

$$\begin{aligned} \alpha &= \frac{h}{L} \qquad \text{where } h = \text{debye-length} = \left(\frac{kT_e}{4\pi n_\infty e^2}\right)^{1/2} \\ \beta &= \frac{D_i}{\epsilon D} \\ \epsilon &= \frac{T_i}{ZT} \\ R &= \frac{\epsilon U_\infty L}{Di} \end{aligned} \tag{2 7}$$

The governing equations now become the following:

$$\alpha^2 \nabla^2 \psi = N_i - N \tag{2 8}$$

$$R(\mathbf{V} \ \nabla N_i) + \nabla \ \gamma_i = 0 \tag{2 9}$$

$$\beta R(\mathbf{V} \ \nabla N) + \nabla \ \gamma = 0 \tag{2 10}$$

$$\gamma_i = N_i \nabla \psi - \epsilon \nabla N_i \tag{2 11}$$

$$\gamma = -N \nabla \psi - \nabla N \tag{2 12}$$

By substituting γ_i and γ_e from Eqs (2 11) and (2 12) into Eqs (2 9) and (2 10), we have

$$R(\mathbf{V} \ \nabla N_i) + \nabla \ (N_i \nabla \psi - \epsilon \nabla N_i) = 0 \tag{2 13}$$

$$\beta R(\mathbf{V} \ \nabla N) - \nabla \ (N \nabla \psi + \nabla N) = 0 \tag{2 14}$$

Eliminating N from Eq (2 14) by using Eq (2 8), we have

$$\beta R(\mathbf{V} \ \nabla N_i) - \nabla \ [N_i \nabla \psi + \nabla N_i] = \beta R \alpha^2 \mathbf{V} \ \nabla(\nabla^2 \psi) - \alpha^2 \nabla \ [\nabla \psi \nabla^2 \psi + \nabla(\nabla^2 \psi)] \tag{2 15}$$

Adding Eqs (2 13) and (2 15), we have

$$(1+\epsilon)\nabla^2 N_i - (1+\beta) R(\mathbf{V} \ \nabla N_i) = \alpha^2 \nabla \ [\nabla \psi \nabla^2 \psi + \nabla(\nabla^2 \psi)] - \beta R \alpha^2 \mathbf{V} \ \nabla(\nabla^2 \psi) \tag{2 16}$$

Multiplying Eq (2 13) by β and subtracting the result from Eq (2 15), we have

$$\nabla \ [(1+\beta) N_i \nabla \psi + (1 - \epsilon\beta)\nabla N_i] = \alpha^2 \nabla \ [\nabla \psi \nabla^2 \psi + \nabla(\nabla^2 \psi)] - \beta R \alpha^2 \mathbf{V} \ \nabla(\nabla^2 \psi) \tag{2 17}$$

Equations (2 16) and (2 17) are exact within the framework of the present theory Since their right-hand sides are identical, then by equating, we recover Eq (2 13):

$$R(\mathbf{V} \ \nabla N_i) = \epsilon \nabla^2 N_i - \nabla \ (N_i \nabla \psi) \tag{2 18}$$

Equation (2 18) may be used in place of either Eqs (2 16) or (2 17)

We shall consider the cases of moderate and low-electric Reynolds number in Sec 8 For the present, let us consider the limiting case of $R \gg 1$ Then Eq (2 18) reduces to

$$\mathbf{V} \ \nabla N_i = 0(R^{-1}) \cong 0$$

and its solution is simply

$$N_i = 1 \tag{2 19}$$

Using this result in Eq (2 17), a complicated equation for ψ is obtained For arbitrary α^2, β, ϵ, and $R\alpha^2$, it is satisfied by

$$\nabla^2 \psi = 0 \tag{2 20}$$

which is the governing equation for ψ in the flow field We shall call this the outer solution and denote it by $\psi = \bar{\psi}$ It is valid away from the body surface The distribution of N_i and N in this outer region are quite uniform, for Eqs (2 20) and (2 8) together imply

$$\lim_{R \to \infty} (N_i - N)/\alpha^2 = 0 \tag{2 21}$$

regardless of the magnitude of α^2

The boundary condition at infinity for ψ is simply $\psi = 0$ To solve Eq (2 20), we need also the value of ψ about the solid body As we shall see presently, there exists an electric boundary layer immediately adjacent to the body surface in which the value of ψ varies rapidly For the moment, let us denote the value of ψ at the outer edge of this boundary layer† by ψ_0, which is distinct from the value of ψ on the body surface, $\psi_B (\neq \psi_0)$ The distribution of ψ_B is, of course, arbitrary and is free to be specified as part of the boundary conditions of the problem The distribution of ψ_0 is, however, unknown and must be found by a detailed boundary-layer analysis.

Note that $\nabla^2 \psi = 0$ does not permit two-dimensional solutions satisfying the requirement that $\psi = 0$ at infinity Hence, the outer solution ψ must be a three-dimensional solution, and the body geometry is important

The boundary layer analysis now proceeds as follows For simplicity, we shall assume quasi two-dimensional flow adjacent to the body surface We let x, y be boundary-layer coordinates‡ with x directed along the body in the direction of flow and y directed normal to the body surface We introduce a scaled normal coordinate η by

$$\eta = yR^{1/2} \tag{2 22}$$

In terms of x and η, Eqs (2 16) and (2 17) become

$$(1+\epsilon)\frac{\partial^2 N_i}{\partial \eta^2} - (1+\beta)\left[u\frac{\partial N_i}{\partial x} + vR^{1/2}\frac{\partial N_i}{\partial \eta}\right] = R\alpha^2 Q \tag{2 23}$$

$$\frac{\partial}{\partial \eta}\left[(1+\beta)N_i \frac{\partial \psi}{\partial \eta} + (1-\epsilon\beta)\frac{\partial N_i}{\partial \eta}\right] = R\alpha^2 Q \tag{2 24}$$

† Subscript 0 shall always indicate conditions at the outer edge of the electric boundary layer Subscript B shall denote conditions on the body surface

‡ In two dimension, boundary-layer coordinates are simply local Cartesian coordinates erected on the surface of the body

where

$$Q = \frac{\partial}{\partial \eta}\left[\frac{\partial \psi}{\partial \eta}\frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^3 \psi}{\partial \eta^3}\right] - \beta\left[u\frac{\partial^3 \psi}{\partial x \partial \eta^2} + vR^{1/2}\frac{\partial^3 \psi}{\partial \eta^3}\right] \quad (2\ 25)$$

Terms of order $O(R^{-1/2})$ and higher have been neglected The product $R\alpha^2$ is assumed finite The velocity components u and v in the x and y directions are of order $O(1)$ and $O(R^{-1/2})$, respectively, as is evident from the basic continuity equation of the neutral gas flow,

$$\nabla \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial (vR^{1/2})}{\partial \eta} = 0 \quad (2\ 26)$$

The governing equations for the electric boundary layer are Eqs (2 23) and (2 24) and are clearly of sixth order in η The appropriate boundary conditions are:

on the body surface,

$$\psi = \psi_B \qquad N_i = 0 \qquad N = N_i - R\alpha^2 \frac{\partial^2 \psi}{\partial \eta^2} \quad (2\ 27)§$$

at $\eta \to \infty$

$$N_i = 1 \qquad N = 1 \qquad \frac{\partial \psi}{\partial \eta} = \frac{1}{R^{1/2}}(\mathbf{n}\ \nabla\tilde{\psi})_0 \quad (2\ 28)$$

where $\mathbf{n}$ is the unit outward normal to the body surface, and $(\mathbf{n}\ \nabla\tilde{\psi})_0$ is the normal derivative of the outer solution $\tilde{\psi}$ at the edge of the electric boundary layer and is assumed finite After the electric boundary layer problem is solved, the value of ψ_0 will come out as part of the solution:

$$\psi_0 = \tilde{\psi}(x, y = 0) = \lim_{\eta \to \infty}\left\{\psi - \eta\frac{(\mathbf{n}\ \nabla\tilde{\psi})_0}{R^{1/2}}\right\} \quad (2\ 29)$$

Knowing ψ_0 about the body, the distribution of $\tilde{\psi}$ in the outer region can then be calculated by standard methods from Eq (2 20)

3 Preliminary Discussion

In the present problem, we have the following dimensionless parameters: α^2, ϵ, β, R, R_e (which is the viscous Reynolds number implicit in the basic velocity profiles), and a presentative value of ψ_B The first step in our analysis was to establish that when $R \gg 1$, an outer solution exists with the properties that the electron and ion number densities are essentially undisturbed and that ψ is harmonic Next we establish that in a thin layer of thickness $O(R^{-1/2})$ adjacent to the body surface an electric boundary layer exists in which electron and ion number densities as well as ψ vary rapidly The governing equations for this electric boundary layer are Eqs (2 23) and (2 24), and they are derived by a straightforward limiting procedure in which R is allowed to tend to infinity while keeping $R\alpha^2$ finite The accuracy of Eqs (2 23) and (2 24) is $O(R^{-1/2})$, and terms of this order originated mainly from the use of boundary layer coordinates These equations still contain three parameters ϵ, β, and $R\alpha^2$ In a realistic situation, the magnitude of R is generally of the same order as R , the viscous Reynolds number of the basic flow The ratio R/R_e is $\epsilon\nu/D_i$ where ν is kinematic viscosity, and ν/D_i is called the Schmidt number The magnitude of ϵ is generally of $O(1)$, but in some cases it may be small Hence, we expect that for $\epsilon \ll 1$ the electric boundary layer will be thicker than the viscous boundary layer The magnitude of β is generally very small, being proportional to

$$[(m_e/m_i)(T_e/T_i)]^{1/2}$$

where m_e and m_i are electron and ion mass, respectively The magnitude of $R\alpha^2$ is the square of the ratio of debye length to the thickness of the electric boundary layer and is a very small number for most situations The magnitude of ψ_B, the potential on the body surface, can often be a large number For example, if $Te = 6000°$ K and $P_B = -10$ v, then $\psi_B \cong 20$ For the present time, we shall assume that the range of ψ_B considered is $O(R^{1/2})$, so that the third boundary condition in Eq (2 28) is considered finite even in the limit of $R \gg 1$

The incompressible assumption on the basic flow is made in the spirit that compressibility effects can be included by standard transformations [3 4] In Sec 4–7, we shall study the structure of the electric boundary layer for $R^{-1/2} \ll 1$ using Eqs (2 23) and (2 24) under the further assumptions that $R\alpha^2 \ll 1$ In Sec 8, we shall outline the appropriate analysis for the entire range of R The $R\alpha^2 \gg 1$ case can also be analyzed straightforwardly, but since it is of no practical interest we shall not be concerned with it

4 Ambipolar Diffusion

When $R\alpha^2 \ll 1$, we may neglect terms involving $R\alpha^2$ in Eq (2 23) to obtain

$$u\frac{\partial N}{\partial x} + vR^{1/2}\frac{\partial N}{\partial \eta} = \frac{1+\epsilon}{1+\beta}\frac{\partial^2 N}{\partial \eta^2} \quad (4\ 1)$$

where N denotes either N_i or N_e in this ambipolar region Note that u and $vR^{1/2}$ are functions of x, $yR^{1/2}$, and $(R/R_e)^{1/2}$, and not of x and η alone unless R/R_e is unity Equation (4 1) is a simple convection-diffusion equation We shall write the boundary conditions as follows:

$$\text{as } \eta \to \infty \quad N = 1 \qquad \text{as } \eta \to \eta^*(x) > 0 \quad N = 0 \quad (4\ 2)$$

Since Eq (4 1) is parabolic, appropriate upstream "initial" condition is also required In Eq (4 2), $\eta^*(x)$ is some arbitrary function indicating the division of the ambipolar diffusion region and a sheath region immediately adjacent to the body surface and is to be determined later After the solution is obtained, we can calculate the quantity

$$I = I(x) = (\partial N/\partial \eta)[x, \eta = \eta^*(x)] \quad (4\ 3)$$

Both I and η^* are assumed to be always positive

In this region charge separation is negligible since from Eq (2 8) we have

$$N = N\ = N_i + O(R\alpha^2) \quad (4\ 4)$$

Neglecting the right-hand side of Eq (2 24) we can integrate once to give

$$(1+\beta)N\frac{\partial \psi}{\partial \eta} + (1-\epsilon\beta)\frac{\partial N}{\partial \eta} = (1+\beta)\frac{(\mathbf{n}\ \nabla\tilde{\psi})_0}{R^{1/2}} \quad (4\ 5)$$

where the constant of integration is determined by fitting boundary condition, Eq (2 28), at $\eta = \infty$ Integrating Eq (4 5), we have simply

$$\psi = \psi_0(x) + \frac{1-\epsilon\beta}{1+\beta}\ln\frac{1}{N} + \frac{(n\ \nabla\tilde{\psi})_0}{R^{1/2}}\left[\eta + \int_\eta^\infty\left(1 - \frac{1}{N}\right)d\eta\right] \quad (4\ 6)$$

§ The values of N_i and N_e on the body surface are of order $O[(l/h\ )(R\alpha^2)^{1/6}]$ where l is the characteristic mean-free-path of the charge particles and is neglected in the present theory (see Sec 9)

Near $\eta \to \eta^*(x)$, the ambipolar solutions for N_i, N_e, and ψ can be written as

$$N_i \cong I(\eta - \eta^*)$$
$$N_e \cong I(\eta - \eta^*) \qquad (4\ 7)$$
$$\frac{\partial\psi}{\partial\eta} \cong \frac{1}{\eta - \eta^*}\left[\frac{(\mathrm{n}\ \nabla\tilde{\psi})_0}{IR^{1/2}} - \frac{1 - \epsilon\beta}{1 + \beta}\right]$$

As $\eta \to \eta^*$, the value of ψ tends to infinity A thin sheath region exists immediately adjacent to the wall Equations (4 7) will serve as boundary conditions to this sheath region

5 Potential Equation for the Sheath

To study the structure of the sheath, we introduce the following transformation:

$$\eta = \eta^*(x) + (R\alpha^2/I)^{1/3}t \qquad (5\ 1)$$
$$N_i = (R\alpha^2 I^2)^{1/3}K(t) \qquad (5\ 2)$$
$$N_e = (R\alpha^2 I^2)^{1/3}G(t) \qquad (5\ 3)$$

By substituting Eqs (5 1) and (5 2) into Eq (2 23), the convection terms on the left-hand side become of order $O[(\eta^*)(R\alpha^2/I)^{2/3}]$ compared with terms on the right-hand side, provided that I, η^*, and their x derivatives are of order unity Equation (2 23) then becomes

$$(1 + \epsilon)\frac{d^2K}{dt^2} = \frac{d}{dt}\left[\frac{d^3\psi}{dt^3} + \frac{d\psi}{dt}\frac{d^2\psi}{dt^2}\right] \qquad (5\ 4)$$

Equation (5 4) can be integrated twice immediately to give

$$(1 + \epsilon)\mathrm{K} = \frac{d^2\psi}{dt^2} + \frac{1}{2}\left(\frac{d\psi}{dt}\right)^2 + C_1(t + C_2) \qquad (5\ 5)$$

where C_1 and C_2 are integrating constants which may be functions of x To fix these constants, we consider the behavior of K and ψ as $t \to \infty$ where they must match with the ambipolar solutions Substituting Eqs (5 1–5 3) into Eqs (4 7) we have

$$K(t \to \infty) = t \qquad (5\ 6a)$$
$$G(t \to \infty) = t \qquad (5\ 6b)$$
$$\frac{d\psi}{dt}(t \to \infty) = \frac{1}{t}\left[\frac{(\mathrm{n}\ \nabla\tilde{\psi})_0}{IR^{1/2}} - \frac{1 - \epsilon\beta}{1 + \beta}\right] \qquad (5\ 6c)$$

Consequently, $C_1 = 1 + \epsilon$ and $C_2 = 0$ Hence, we have

$$(1 + \epsilon)(K - t) = \frac{1}{2}\left(\frac{d\psi}{dt}\right)^2 + \frac{d^2\psi}{dt^2} \qquad (5\ 7)$$

At this point, we need another equation between K and ψ This is obtained by eliminating Q between Eqs (2 23) and (2 24) and writing the resulting equation in the new variables:

$$(1 + \epsilon)\frac{d^2K}{dt^2} = \frac{d}{dt}\left[(1 + \beta)K\frac{d\psi}{dt} + (1 - \epsilon\beta)\frac{dK}{dt}\right]$$

that can again be integrated once to give

$$\epsilon\frac{dK}{dt} = K\frac{d\psi}{dt} + \frac{1 + \epsilon}{1 + \beta} - \frac{(\mathrm{n}\ \nabla\tilde{\psi})_0}{IR^{1/2}} \qquad (5\ 8)$$

where the constant of integration has been chosen to satisfy boundary conditions in Eqs (5 6) Solving for K from Eq (5 7) and substituting into Eq (5 8), we have finally

$$\epsilon\frac{d^3\psi}{dt^3} - (1 - \epsilon)\frac{d\psi}{dt}\frac{d^2\psi}{dt^2} - \frac{1}{2}\left(\frac{d\psi}{dt}\right)^3 - (1 + \epsilon)\,t\frac{d\psi}{dt} = (1 + \epsilon)X \qquad (5\ 9)$$

where

$$X = \frac{1 - \epsilon\beta}{1 + \beta} - \frac{(\mathrm{n}\ \nabla\tilde{\psi})_0}{IR^{1/2}}$$

This is the same equation studied by Cohen [2]

6 Sheath Solution

Since ψ itself does not appear in Eq (5 9), we can lower the order of the equation by defining

$$W = -(d\psi/dt) \qquad (6\ 1)$$

Equation (5 9) then becomes

$$\epsilon\frac{d^2W}{dt^2} = (\epsilon - 1)W\frac{dW}{dt} + \frac{W}{2}[W^2 + 2(1 + \epsilon)t] - (1 + \epsilon)X \qquad (6\ 2)$$

The boundary conditions are as follows On the body surface, we require that

$$K = 0 \qquad (6\ 3)$$
$$G = 0 \qquad (6\ 4)$$

Since $K - G = -dW/dt$, we have at $\eta = 0$

$$dW/dt(t = t_B) = 0 \qquad (6\ 5)$$

where t_B is the value of t at $\eta = 0$ and is given by

$$t_B = -\eta^*(I/R\alpha^2)^{1/3} \qquad (6\ 6)$$

from Eq (5 1) From Eq (5 7), since $K = 0$ and $dW/dt = 0$ at $t = t_B$, we have

$$W_B^2 = -2(1 + \epsilon)t_B \qquad (6\ 7)$$

Equation (6 7) states that the wall lies somewhere on a parabola in the (W,t) plane At $t \to \infty$, we require that the sheath solutions match with the ambipolar solutions given in Eqs (4 7):

$$W(t \to \infty) = X/t \qquad (6\ 8)$$

Equations (6 5, 6 7, and 6 8) are the complete boundary conditions for Eq (6 2)

Although no rigorous proof is offered, the solution $W(t)$ is considered unique and is completely specified by the values of ϵ and X ¶ In other words, for a given pair of ϵ and X, only one single solution can satisfy Eq (6 2) and its boundary conditions (6 5, 6 7, and 6 8) Hence, t_B is some universal function of ϵ and X and can be calculated once and for all

After $W(t;\epsilon,X)$ has been obtained, the distribution of ψ in the sheath is given by

$$\psi = \psi_B(x) - \int_{t_B(\epsilon\ X)}^{t} W(t;\epsilon,X)dt \qquad (6\ 9)$$

As $t \to \infty$, this must be matched with the ambipolar diffusion solution near $\eta = \eta^*$ Since Eq (6 9) diverges at $t = \infty$, we first rewrite it for large t as

$$\psi = \psi_B - X\ln t - F(\epsilon,X) \qquad (6\ 10a)$$

where

$$F(\epsilon,X) = \int_{t_B(\epsilon\ X)}^{1} W dt + \int_1^\infty \left(W - \frac{X}{t}\right)dt \qquad (6\ 10b)$$

The function $F(\epsilon, X)$ is a universal function of ϵ and X Figure 1 is a plot of F vs X for $\epsilon \ll 1$ obtained to slide rule accuracy from Cohen's numerical solutions An important property of F is that $F \to \infty$ as $X \to 1$ and $F \to -\infty$ as

¶ For the special case $\epsilon = 0$, an iso cline study of the simplified, first-order equation for W gives convincing support to the uniqueness claim

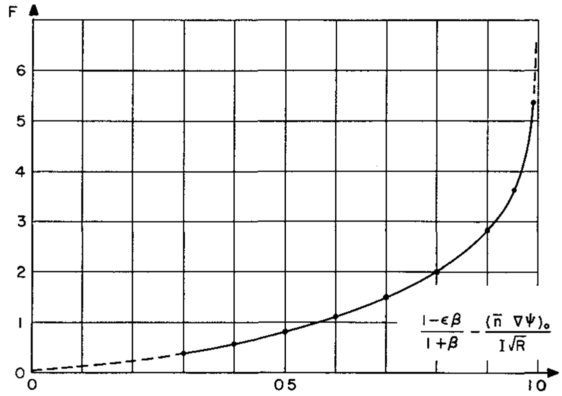

Fig 1 F vs $(1 - \epsilon\beta/1 + \beta)$ — $[(n \cdot \Delta\tilde{\psi})_0/IR^{1/2}]$ for $\epsilon \ll 1$

$X \rightarrow -\epsilon$ For $\epsilon = 1$, the curve F vs X is symmetric about the origin

The ambipolar diffusion solution, Eq (4 6), is now written as, for $\eta \cong \eta^*$,

$$\psi = \psi_0(x) - X \ln t - \frac{1}{3} X \ln(R\alpha^2 I^2) + \frac{(n \cdot \nabla\tilde{\psi})_0}{I\,R^{1/2}}\,[1 + D] \quad (6.11)$$

where

$$D = I\left[\int_0^{\frac{1}{I}}\left(\frac{1}{I\eta} - \frac{1}{N}\right)d\eta + \int_{\frac{1}{I}}^{\infty}\left(1 - \frac{1}{N}\right)d\eta\right]$$

Comparing Eqs (6 10a) and (6 11), we have

$$\psi_c = \psi_B - \Delta\psi \quad (6\ 12)$$

with

$$\Delta\psi = F(\epsilon,X) + \frac{1}{3} X \ln\frac{1}{R\alpha^2 I^2} + \frac{(n \cdot \nabla\tilde{\psi})_0}{IR^{1/2}}\,(1 + D) \quad (6\ 13)$$

It is clear that $\Delta\psi$ is the potential difference between ψ_B and ψ_0 If $t_B(\epsilon,X)$ is finite and of order unity, then from Eq (6 6) we see that $\eta^* = O(R\alpha^2)^{1/3}$ This is the thickness of the sheath and is seen to be small Hence, in the calculation of the ambipolar diffusion solution for N, η^* can be set to zero, thus greatly simplifying matters For this case, I is independent of either X or $R\alpha^2$ If t_B is very large,** then Eq (6 6) must be used to supplement the ambipolar calculation, and I itself will depend on X and $R\alpha^2$ besides the basic flow velocity profiles Note that, in the special limiting case of $(R/R_e)^{1/2} \ll 1$, the viscous boundary layer is much thinner than the ambipolar diffusion layer For this case Eq (4 1) reduces to

$$\partial N/\partial\xi = \partial^2 N/\partial\eta^2 \quad (6\ 14)$$

where

$$\xi = \frac{1 + \beta}{1 + \epsilon}\int^x \frac{dx}{U(x)}$$

and $U(x)$ is the inviscid basic flow velocity evaluated at the edge of the viscous boundary layer Eq (6 14) can be solved exactly by a variety of methods, and if $\eta^* \cong 0$ then I will be a function of ξ only

For a given flow of weakly ionized gas, $\Delta\psi$ is mainly controlled by the values of ϵ and X and is only weakly dependent on $R\alpha^2 I^2$ since this group of parameters appears logarithmically The magnitude of X, however, depends on the outer solution via the term $(n \cdot \nabla\tilde{\psi}_0)/I(R)^{1/2}$ For finite ψ_B and asymptotically large R, X will reduce to simply $(1 - \epsilon\beta)/(1 + \beta)$, and for this case $\Delta\psi$ will be independent of the outer solution It should be noted that the detailed velocity profiles, and therefore the value of R_e, affects only the value of I Hence, as far as $\Delta\psi$ is concerned, they have very small effects

Figures 2a and 2b show qualitatively the number densities and potential distributions in all three regions In Fig 2b, several cases are shown to illustrate the effects of ψ_B on the outer region The number density distributions are independent of ψ_B except in the sheath region

7 Current-Voltage Characteristics

The normal surface current per unit area is, by definition, given by

$$J_B = e[n \cdot (\mathbf{\Gamma} - Z\mathbf{\Gamma}_i)]_{y=0} \quad (7\ 1)$$

$$= \frac{eD\,n_{e_\infty}}{L}\,[n \cdot (\gamma - \beta\gamma_i)]_{y=0}$$

From simple kinetic theory, the electrical conductivity of the gas is given by[5]

$$\sigma = (1 + \beta)\,\frac{e^2 D_e n_{e_\infty}}{kT_e} \quad (7\ 2)$$

We can thus define a characteristic current density J_∞ by

$$J_\infty = \sigma\,\frac{kT_e}{eL} = (1 + \beta)\,\frac{D_e n_{e_\infty} e}{L} \quad (7\ 3)$$

Using this expression in Eq (7 1), we have

$$\frac{J_B}{J_\infty} = \frac{1}{1 + \beta}\,[n \cdot (\gamma - \beta\gamma_i)]_{y=0} \quad (7\ 4)$$

Under the approximations that $\beta R^{3/2}\alpha^2 \ll 1$ and $R^{-1/2} \ll 1$, it can be shown[6] that it is consistent to evaluate Eq (7 4) at the edge of the ambipolar region instead of on the body surface Physically, this means that the current flow inside the electric boundary layer is essentially normal to the body surface Since $N_i \cong N_e \cong 1\ 0$ at the edge of the ambipolar region, we have from Eqs (2 11) and (2 12) the following expression for J_B/J_∞:

$$\frac{J_B}{J_\infty} \cong -\left(\frac{\partial\tilde{\psi}}{\partial y}\right)_{y=y_0} = -(n \cdot \nabla\tilde{\psi})_0 \quad (7\ 5)$$

We now write, following Ref 6,

$$j_3 = -\frac{1}{\psi_0}\left(\frac{\partial\tilde{\psi}}{\partial y}\right)_{y_0} = -\frac{(n \cdot \nabla\tilde{\psi})_0}{\psi_0} \quad (7\ 6)$$

Since the outer solution satisfies the Laplace equation, j_3 is

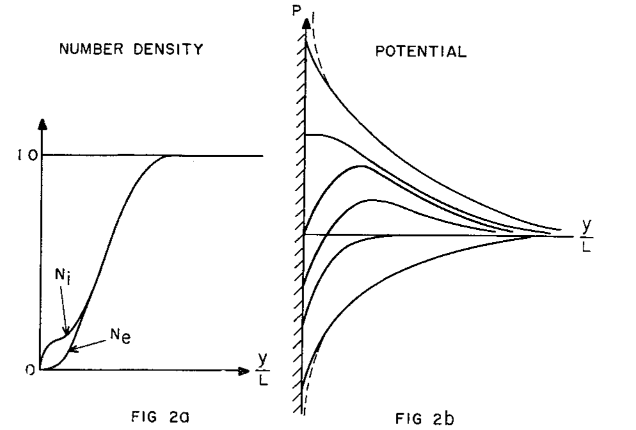

Fig 2 Number density and potential distributions

** A detailed analysis will show that, for small $1 - X$, t_b is $O[\ln(1/1 - X)]^{2/3}$ and F is $O[\ln(1/1 - X)]$

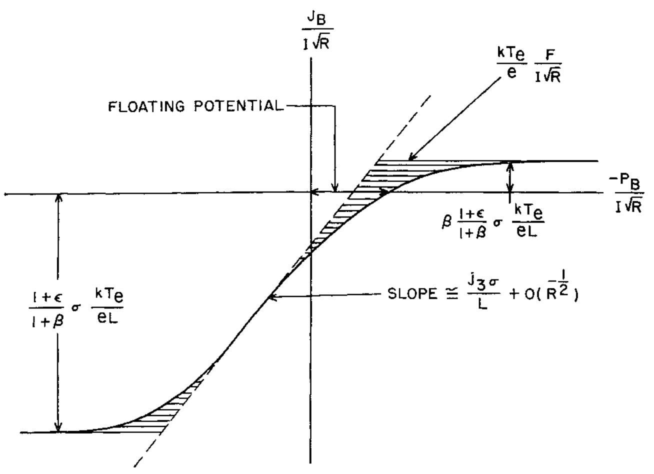

Fig 3 Current-voltage characteiistic

$O(1)$ and is generally positive Using Eq (7 5) in the definition of X and Eq (7 6) and substituting the results in Eqs (6 12) and (6 13), we have

$$\psi_B = \frac{1}{j_3}\frac{J_B}{J_\infty} + \left\{\frac{1}{3}\left[\frac{1-\epsilon\beta}{1+\beta} + \frac{1}{IR^{1/2}}\frac{J_B}{J_\infty}\right]\ln\frac{1}{R\alpha^2 I^2} - \frac{1}{IR^{1/2}}\frac{J_B}{J_\infty}(1+D)\right\} + F\left(\epsilon, \frac{1-\epsilon\beta}{1+\beta} + \frac{1}{IR^{1/2}}\frac{J_B}{J_\infty}\right) \quad (7\ 7)$$

which relates J_B with ψ_B and is, therefore, the current-voltage characteristic desired See Fig 3

The first term in Eq (7 7) represents the potential drop in the outer region The term in braces represents the potential drop in the ambipolar diffusion region The last term, $F(\epsilon, X)$, represents the potential drop inside the sheath Several important conclusions can be deduced from Eq (7 7) They are as follows:

1) The floating potential is simply obtained by setting $J_B = 0$ We have

$$-P_B\ (\text{floating}) = \frac{kT_e}{e}\left\{\frac{1}{3}\frac{1-\epsilon\beta}{1+\beta}\ln\frac{1}{R\alpha^2 I^2} + F\left(\epsilon, \frac{1-\epsilon\beta}{1+\beta}\right)\right\} \quad (7\ 8)$$

Hence, a plot of P_B (floating) vs $\ln(R\alpha^2 I^2)$ will yield a slope equal to $\frac{1}{3}(kT/e)[1 + 0(\beta)]$ Since β is generally very small, this affords a simple way of measuring T

2) For $J_B/J_\infty = O(1)$ and $R^{-1/2} \ll 1$, Eq (7 7) reduces to

$$-P_B \cong \frac{1}{j_3}\frac{\sigma}{L}J_B + \frac{kT_e}{e}\left\{\frac{1}{3}\frac{1-\epsilon\beta}{1+\beta}\ln\frac{1}{R\alpha^2 I^2} + F\left(\epsilon, \frac{1-\epsilon\beta}{1+\beta}\right)\right\} = \frac{1}{j_3}\frac{\sigma}{L}J_B - P_B(\text{floating}) \quad (7\ 9)$$

Hence, in this range of J_B, the current-voltage characteristic is linear, and its slope is directly related to the freestream electrical conductivity

3) Since $F \to \pm\infty$ as $X \to 1$ and $X \to -\epsilon$, the saturation currents are given by

$$J_B = J_\infty I R^{1/2}\frac{\beta(1+\epsilon)}{1+\beta} \qquad \psi_B \to \infty$$

$$J_B = J_\infty I R^{1/2}\frac{1+\epsilon}{1+\beta} \qquad \psi_B \to -\infty$$

Note that

$$J_\infty I R^{1/2} = \sigma\frac{kT}{e} I \left(\frac{\epsilon U_\infty}{D_i L}\right)^{1/2}$$

4) The ratio of saturation currents is then seen to be β Note that the original classical low-density theory for Langmuir probes predicts that this ratio be $\epsilon\beta$ instead of β

8 Arbitrary Electric Reynolds Numbei

If R is finite, the outer solution loses its identity In this section, we shall briefly outline the appropriate analysis for the entire range of R, but $\alpha^2 \ll 1$ is always assumed

Low R Case, $R \ll 1$

When $R \ll 1$ so that terms of order $O(R)$ can be neglected, Eqs (2 16) and (2 17) reduce to

$$\nabla^2 N_i = \frac{\alpha^2}{1+\epsilon}\nabla\ [\nabla\psi\nabla^2\psi + \nabla(\nabla^2\psi)] \quad (8\ 1)$$

$$\nabla\ [(1+\beta)N_i\nabla\psi + (1-\epsilon\beta)\nabla N_i] = \alpha^2\nabla\ [\nabla\psi\nabla^2\psi + \nabla(\nabla^2\psi)] \quad (8\ 2)$$

For a sphere the problem reduces identically to that studied by Su and Lam[1] and Cohen [2] It is clear that the generalization to an arbitrary body is immediate In the region away from the body surface, the physical process involved is only ambipolar diffusion We let $\alpha^2 \to 0$ in Eqs (2 8, 8 1, and 8 2) to obtain

$$N_i \cong N = N \quad (8\ 3)$$

$$\psi = \frac{1-\epsilon\beta}{1+\beta}\ln\frac{1}{N} + \psi_1$$

where

$$\nabla^2 N = 0 \quad (8\ 4)$$

$$\nabla\ [N\nabla\psi_1] = 0 \quad (8\ 5)$$

The appiopriate boundary conditions are

$$N = 1 \qquad \text{at infinity} \quad (8\ 6)$$

$$N = 0 \qquad |\nabla\psi_1| \to \infty \qquad \text{at the edge of the sheath}$$

Denoting by y the local normal coordinate measured from the sheath edge, then for small y we can write the ambipolar solutions as

$$N = iy$$

$$\frac{\partial\psi_1}{\partial y} = \frac{1}{iy}\left[\lambda_1 - \frac{1-\epsilon\beta}{1+\beta}\right] \quad (8\ 7)$$

where $i = (\mathbf{n}\ \nabla N)_{y=0}$ and is now known, and λ_1 is yet some arbitrary function of position on the body surface The analysis of the sheath can now proceed in a similar manner as that of Cohen's [2] The distribution of λ controls the current and therefore depends on ψ_B and is determined by matching the outer ambipolar diffusion solution with the sheath solution

Modeiate R Case, $R = 0(1)$

For moderate R, we again consider the limit of $\alpha \to 0$ for Eqs (2 16) and (2 17) The outer region exists but is not distinct from the ambipolar region We have for this region

$$N_i \cong N = N$$

$$\psi = \frac{1-\epsilon\beta}{1+\beta}\ln\frac{1}{N} + \psi_2 \quad (8\ 8)$$

where

$$R(\mathbf{V}\ \nabla N) = \frac{1+\epsilon}{1+\beta}\nabla^2 N$$

$$\nabla\ [N\nabla\psi_2] = 0 \quad (8\ 9)$$

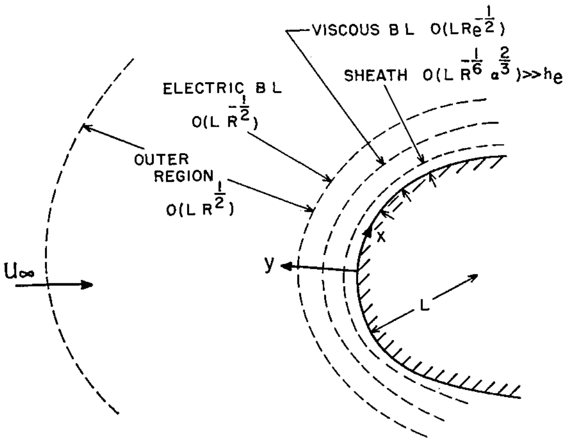

Fig 4 Regions of electrical disturbances

with boundary conditions identical to Eqs (8 6) The sheath analysis that follows is again identifical to Cohen's

9 Discussion

In a general problem where $R\alpha^2 \ll 1$ is satisfied, the disturbances can be qualitatively described as follows Immediately adjacent to the body, there is a sheath From the present continuum theory, the thickness of the sheath y_{sh} is

$$y_{sh} \cong (L/R^{1/2})(R\alpha^2)^{1/3}t_B$$

or

$$y_{sh} \cong h_e R^{-1/6}(L/h)^{1/3}t_B \qquad (9\ 1)$$

and the potential drop across the sheath is $F(\epsilon, X)$ We see from Eq (9 1) that $y_{sh} \gg h_e$ in general Next to this sheath there is an ambipolar diffusion region of thickness

$$y_{\text{ambipolar}} = LR^{-1/2} \qquad (9\ 2)$$

across which the potential drop is given by the braces in Eq (7 7) For $\epsilon \ll 1$, the ambipolar diffusion region is generally thicker than the thickness of the viscous boundary layer External to the ambipolar diffusion region, the charge number densities are rigidly maintained uniform by convection, and in this outer region simple electrical conduction prevails The extent of this region is $O(J_B L/J_\infty)$ Thus, at highly negative probe potentials, the electrical field penetrates far into the outer region, the extent being $O(\beta I R^{1/2} L)$ At highly positive probe potentials, the extent would be $O(IR^{1/2}L)$ If the body in question is used as a probe, we see that such extensive disturbances created by the probe are quite undesirable (See Fig 4)

In order to use Eq (7 7), the current-voltage characteristic, one must first calculate the quantities j_3, D, and I To illustrate the procedure, let us assume that J_B/J_∞ is specified as a function of x on the body surface We can then solve the Laplace equation, Eq (2 20), by any standard methods using Eq (7 5) as boundary conditions The value of $j_3(x)$ is simply obtained from Eq (7 6) Next we solve the simple convection diffusion problem, Eq (4 1), by any standard methods and can as a first approximation take $\eta^* = 0$ in the boundary conditions, Eq (4 2) Once N is known, the values of $D(x)$ and $I(x)$ follow from their definitions, and $\psi_B(x)$ follows from Eq (7 7) This process is to be repeated for every J_B/J_∞ assumed

If the estimated sheath thickness, Eq (9 1), is much smaller than the estimated mean-free-path of the charged particles, then the continuum sheath analysis given in Secs 5 and 6 would not apply However, the correct current-voltage characteristic would still be given by Eq (7 7) if we now interpret $F(\epsilon X)$ as the appropriate noncontinuum sheath potential drop Actually, for this noncontinuum sheath F must depend on an additional parameter $K = R^{1/6}(he/L)^{1/3}(l/h_e)$, which is the ratio of a characteristic charge-neutral mean-free-path to the sheath thickness In other words, $F = F(\epsilon, X, K)$ When $K \to \infty$, the sheath is collisionless At the present time, of course, there is no available theory for this function, but it appears possible to construct this function by a series of controlled experiments Note from Eq (9 1) that the sheath is much thicker than h The continuum sheath assumption is then formally valid when $l/y_{sh} \ll 1$ In other words, we require

$$K = R^{1/6}(h_e/L)^{1/3}(l/h) \ll 1 \qquad (9\ 3)$$

If Eq (9 3) is not satisfied, then the boundary conditions for N_i and N_e at the body surface should be modified in a similar manner as slip effects in low-density gas dynamics In other words,

$$N(x,\eta = \eta^*)\alpha K(\partial N/\partial n)(x,\eta^*) = KI$$

It is interesting to note that Eq (9 3) can always be satisfied for a given plasma if L is made sufficiently large

An interesting feature of the structure of the potential distribution about the body is that for moderate body potentials, such as $\psi_B(\text{floating}) > \psi_B \cong 0$, the body is surrounded by a positive potential ring when $R^{-1/2} \ll 1$ This positive potential ring decays in strength as R is decreased, thus it is not expected to be present in any noncontinuum theory Thus, if such a ring is detected in an experiment, it would serve as a good indication that the present continuum theory instead of noncontinuum theory should be used To the author's knowledge, no other theory, continuum or otherwise, predicts such a positive potential ring

In the works of Chung[3] and Talbot,[4] the existence of the outer solution was not appreciated In Talbot's model, the sheath was assumed to consist of a free-fall region, and apparently the potential drop was assumed to occur entirely in the sheath This assumption is not supported by the present work

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